A NOTE ON THE IDEAL $I[S;\lambda]$

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Abstract. Shelah asked whether GCH implies that $\diamondsuit(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$ holds for every singular cardinal. We give an affirmative answer for every singular cardinal λ of uncountable cofinality for which $\{\mu<\lambda\mid \square^*_\mu \text{ holds}\}$ is stationary in λ.

The proof builds on a recent paper by Levine and goes through showing that for every singular strong limit cardinal λ of uncountable cofinality such that $\{\mu<\lambda\mid \square^*_\mu \text{ holds}\}$ is stationary in λ, for every stationary $S\subset\lambda^+$, $I[S;\lambda]=I[\lambda^+]\restriction\mathrm{Tr}(S)$. This stands in contrast to a model of Gitik and Rinot in which GCH holds and the last equality fails for $\lambda=\aleph_\omega$ and some stationary $S\subset\aleph_{\omega+1}$.

1. Introduction

Hereafter, λ denotes a singular cardinal. A longstanding open problem of Shelah is whether GCH implies that $\diamondsuit(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$ holds. By [She84], an affirmative answer follows from $\square^*_\lambda$.[1]

A few days ago, in [Lev22, §2.1], Levine published a compactness theorem for $\square^*_\lambda$, proving that if λ is a singular strong limit cardinal of uncountable cofinality, and $\{\mu<\lambda\mid \square^*_\mu \text{ holds}\}$ is stationary, then the existence of a good scale at a particular product of cardinals implies that $\square^*_\lambda$ holds. The purpose of this note is to prove that an affirmative answer to Shelah's question follows without the good scale hypothesis.

Theorem A. *Suppose that λ is a singular strong limit cardinal of uncountable cofinality and $\{\mu<\lambda\mid \square^*_\mu \text{ holds}\}$ is stationary. Then $2^\lambda=\lambda^+$ iff $\diamondsuit(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$ holds.*

The proof makes use of our previous paper [Rin10]. In that paper, for every stationary $S\subset\lambda^+$, an ideal $I[S;\lambda]$ over $\mathrm{Tr}(S):=\{\delta\in E^{\lambda^+}_{>\omega}\mid S\cap\delta \text{ is stationary in } \delta\}$ was introduced, and Shelah's result was factored through this ideal, as follows.

Fact 1 ([Rin10]).
1. *If $\square^*_\lambda$ holds, then $I[S;\lambda]$ contains a stationary set for every stationary $S\subset\lambda^+$ that reflects stationary often;*
2. *For every stationary $S\subset\lambda^+$ such that $I[S;\lambda]$ contains a stationary set, $2^\lambda=\lambda^+$ iff $\diamondsuit(S)$ holds;*
3. *Assuming the consistency of a supercompact cardinal, it is consistent that $\square^*_\lambda$ fails, and yet, $I[S;\lambda]$ contains a stationary set for every stationary $S\subset\lambda^+$ that reflects stationary often.*

Here, via minor adjustments to Levine's argument from [Lev22], it is proved:

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[1]All missing definitions may be found in Subsection 1.1 below.

Theorem B. *Suppose that λ is a singular strong limit cardinal of uncountable cofinality and $\{\mu < \lambda \mid \square^*_\mu \text{ holds}\}$ is stationary in λ.*

For every stationary $S \subset \lambda^+$, $I[S;\lambda] = I[\lambda^+] \restriction \mathrm{Tr}(S)$.

The point is that by a celebrated theorem of Shelah [She93], $I[\lambda^+] \restriction E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ indeed contains a stationary set, and hence Theorem A follows from Theorem B together with Fact 1. To compare, by [GR12, Theorem A], it is relatively consistent with the existence of a supercompact cardinal that GCH holds, $I[\aleph_{\omega+1}] = \mathcal{P}(\aleph_{\omega+1})$, and there exists a stationary $S \subseteq E^{\aleph_{\omega+1}}_\omega$ that reflects stationarily often, and yet, $I[S;\aleph_\omega]$ contains no stationary set. Note that GCH implies that $\square^*_{\aleph_n}$ holds for all $n < \omega$.

1.1. **Notation and definitions.** For a set of ordinals A, we write $\mathrm{acc}^+(A) := \{\alpha < \sup(A) \mid \sup(A \cap \alpha) = \alpha > 0\}$ and $\mathrm{acc}(A) := A \cap \mathrm{acc}^+(A)$. For two nonempty sets of ordinals A, B, we write $A \subseteq^* B$ to mean that $\sup(A \setminus B) < \sup(B)$.

For a pair of infinite regular cardinals $\theta < \kappa$, let $E^\kappa_\theta := \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \theta\}$, and define $E^\kappa_{\leq\theta}$, $E^\kappa_{<\theta}$, $E^\kappa_{\geq\theta}$, $E^\kappa_{>\theta}$, $E^\kappa_{\neq\theta}$ analogously. For a stationary subset $S \subseteq \kappa$, Jensen's diamond principle $\diamondsuit(S)$ asserts the existence of a sequence $\langle A_\gamma \mid \gamma \in S\rangle$ such that, for every subset $A \subseteq \kappa$, for stationarily many $\gamma \in S$, $A_\gamma = A \cap \gamma$ (cf. [Rin11]). For an infinite cardinal μ, Jensen's weak square principle $\square^*_\mu$ asserts the existence of a matrix $\langle C_{\delta,i} \mid \delta < \mu^+, i < \mu\rangle$ such that for all $\delta < \mu^+$ and $i < \mu$:

1. $C_{\delta,i}$ is a closed subset of δ with $\sup(C_{\delta,i}) = \sup(\delta)$ and $\mathrm{otp}(C_{\delta,i}) \leq \kappa$;
2. for every $\gamma \in \mathrm{acc}(C_{\delta,i})$, there exists $j < \mu$ such that $C_{\delta,i} \cap \gamma = C_{\gamma,j}$.

In case that μ is a singular cardinal, one can moreover demand that $\mathrm{otp}(C_{\delta,i})$ be strictly smaller than μ for all $\delta < \mu^+$ and $i < \mu$.

2. *pcf* SCALES

Throughout this section, λ denotes a singular cardinal *of uncountable cofinality*, and $\vec{\lambda} = \langle \lambda_i \mid i < \mathrm{cf}(\lambda)\rangle$ is a strictly increasing sequence of regular cardinals, converging to λ. We start by recalling a few rudimentary concepts from *pcf* theory.

Definition 2.1. Let $f, g \in \prod \vec{\lambda}$.

- For an ordinal $i < \mathrm{cf}(\lambda)$, we write $f <^i g$ iff $f(j) < g(j)$ for all $j \in \mathrm{cf}(\lambda) \setminus i$;
- We write $f <^* g$ to express that $f <^i g$ for some $i < \mathrm{cf}(\lambda)$.

Definition 2.2. A sequence $\vec{f} = \langle f_\delta \mid \delta < \lambda^+\rangle$ is said to be a *scale in* $\prod \vec{\lambda}$ iff all of the following hold:

- for every $\delta < \lambda^+$, $f_\delta \in \prod \vec{\lambda}$;
- for all $\gamma < \delta < \lambda^+$, $f_\gamma <^* f_\delta$;
- for every $g \in \prod \vec{\lambda}$, there exists $\delta < \lambda^+$ such that $g <^* f_\delta$.

Fact 2.3 (Shelah, [She94, Claim 2.1])**.** *For every singular cardinal λ of uncountable cofinality, there exists a strictly increasing and continuous sequence $\langle \mu_i \mid i < \mathrm{cf}(\lambda)\rangle$ of cardinals, converging to λ, such that $\prod_{i<\mathrm{cf}(\lambda)} \mu_i^+$ admits a scale.*

Definition 2.4. Suppose $\vec{f} = \langle f_\delta \mid \delta < \lambda^+\rangle$ is a scale in $\prod \vec{\lambda}$. Let $\delta \in \mathrm{acc}(\lambda^+)$.

- δ is *good* (with respect to $\vec{f}$) iff there exist a cofinal subset $A \subseteq \delta$ and $i < \mathrm{cf}(\lambda)$ such that, for every pair of ordinals $\gamma < \delta$ from A, $f_\gamma <^i f_\delta$;
- $h \in \prod \vec{\lambda}$ is an *exact upper bound* for $\vec{f} \restriction \delta$ iff the two hold:

(1) for every $\gamma < \delta$, $f_\gamma <^* h$;
(2) for every $g \in \prod \vec{\lambda}$ such that $g <^* h$, there exists $\gamma < \delta$ such that $g <^* f_\gamma$.

Remark 2.5. (1) Every $\delta \in \mathrm{acc}(\lambda^+)$ of cofinality $< \mathrm{cf}(\lambda)$ is good;
(2) An ordinal $\delta \in E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ is good iff $\vec{f} \restriction \delta$ admits an exact upper bound.

Following Cummings, Foreman and Magidor [CFM01, Definition 3.9], we say that a scale $\vec{f} = \langle f_\delta \mid \delta < \lambda^+ \rangle$ is *continuous* iff for every $\delta \in \mathrm{acc}(\lambda^+)$ such that $\vec{f} \restriction \delta$ admits an exact upper bound, f_δ is an exact upper bound.

In [Lev22, §2.1], Levine introduced the concept of *totally continuous* scales. A natural weakening of which reads as follows.

Definition 2.6 (Aligned scales)**.** We say that a scale $\vec{f} = \langle f_\delta \mid \delta < \lambda^+ \rangle$ is *aligned* iff for every $\delta \in \mathrm{acc}(\lambda^+)$, all of the following hold:

- if $\mathrm{cf}(\delta) < \mathrm{cf}(\lambda)$, then for every cofinal $B \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$, for all but boundedly many $i < \mathrm{cf}(\lambda)$,
$$f_\delta(i) = \sup\{f_\beta(i) \mid \beta \in B\};$$
- if $\mathrm{cf}(\delta) = \mathrm{cf}(\lambda)$, then for every club $B \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$, for club many $i < \mathrm{cf}(\lambda)$,
$$f_\delta(i) = \sup\{f_\beta(i) \mid \beta \in B, \mathrm{otp}(B \cap \beta) < i\};$$
- if $\mathrm{cf}(\delta) > \mathrm{cf}(\lambda)$ is good, then for every cofinal $A \subseteq \delta$, there exists a cofinal $B \subseteq A$ such that for all but boundedly many $i < \mathrm{cf}(\lambda)$, $\langle f_\beta(i) \mid \beta \in B \rangle$ is strictly increasing and converging to $f_\delta(i)$. In particular, f_δ is an exact upper bound of $\vec{f} \restriction \delta$.

Remark 2.7. Levine's notion of 'totally continuous' scale is the conjunction of a scale being aligned and all of its points of cofinality $> \mathrm{cf}(\lambda)$ being good.

A combination of Fact 2.3 with Levine's proof of [Lev22, Lemma 2.6] yields the following:

Fact 2.8 (Shelah + Levine)**.** *For every singular cardinal λ of uncountable cofinality, there exists a strictly increasing and continuous sequence $\langle \mu_i \mid i < \mathrm{cf}(\lambda) \rangle$ of cardinals, converging to λ, such that $\prod_{i<\mathrm{cf}(\lambda)} \mu_i^+$ admits an aligned scale.*

3. A relative of the approachability ideal

Definition 3.1 ([Rin10, Definition 2.4])**.** For a singular cardinal λ and a proper subset $S \subset \lambda^+$, $I[S;\lambda]$ stands for the collection of all subsets $T \subseteq \mathrm{Tr}(S)$ for which there exist a club $C \subseteq \lambda^+$ and a coloring $d : [\lambda^+]^2 \rightarrow \mathrm{cf}(\lambda)$ such that all of the following hold:

(1) d is locally small, that is, for every $\gamma < \lambda^+$ and every $i < \mathrm{cf}(\lambda)$,
$$|\{\alpha < \gamma \mid d(\alpha, \gamma) \leq i\}| < \lambda;$$
(2) d is subadditive of the first kind, that is, for all $\alpha < \beta < \gamma < \lambda^+$,
$$d(\alpha, \gamma) \leq \max\{(\alpha, \beta), d(\beta, \gamma)\};$$
(3) for every $\delta \in T \cap C \cap E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$, there exists a stationary $S_\delta \subseteq S \cap \delta$ with $\sup(d``[S_\delta]^2) < \mathrm{cf}(\lambda)$.

To compare, Shelah's *weak approachability ideal* $I[\lambda^+;\lambda]$ stands for the collection of all subsets $T\subseteq\lambda^+$ for which there exist a club $C\subseteq\lambda^+$ and a coloring $d:[\lambda^+]^2\rightarrow\mathrm{cf}(\lambda)$ satisfying Clauses (1) and (2) above together with the following:

(3') for every $\delta\in T\cap C\cap E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$, there exists a *cofinal* $X_\delta\subseteq\delta$ such that $\sup(d``[X_\delta]^2)<\mathrm{cf}(\lambda)$.

Remark 3.2. We omit the definition of Shelah's *approachability ideal* $I[\lambda^+]$ and settle for stating that if λ is a strong limit, then $I[\lambda^+;\lambda]$ and $I[\lambda^+]$ coincide (see [Eis10, Proposition 3.23]).

Remark 3.3. For every $S\subset\lambda^+$, $I[S;\lambda]\subseteq I[\lambda^+;\lambda]\restriction\mathrm{Tr}(S)$. By [Rin10, Corollary 2.10], if $S\subseteq E^{\lambda^+}_{\neq\mathrm{cf}(\lambda)}$, then $I[S;\lambda]=I[\lambda^+;\lambda]\restriction\mathrm{Tr}(S)$.

The next result shows it is consistent that for a singular strong limit of *countable* cofinality, for some stationary $S\subset\lambda^+$, $I[S;\lambda]\neq I[\lambda^+;\lambda]\restriction\mathrm{Tr}(S)$.

Fact 3.4 ([GR12, Theorem A])**.** *Assuming the consistency of a supercompact cardinal, it is consistent that* GCH *holds,* $I[\aleph_{\omega+1}]=\mathcal{P}(\aleph_{\omega+1})$*, and yet, for some stationary* $S\subseteq E^{\aleph_{\omega+1}}_\omega$ *that reflects stationarily often,* $I[S;\aleph_\omega]$ *contains no stationary set.*

Our proof of Theorem B will make use of the following characterization of $I[S;\lambda]$.

Fact 3.5 ([Rin10, Proposition 3.17])**.** *Suppose that λ is a singular cardinal, and* $S\subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$*. A subset* $T\subseteq\mathrm{Tr}(S)$ *is in* $I[S;\lambda]$ *iff there exist collections* $\{A_\delta\mid\delta\in T\}$ *and* $\{B^i_\gamma\mid\gamma\in S,i<\lambda\}\subseteq[\lambda^+]^{<\lambda}$ *such that for club many* $\delta\in T$*:*

(1) $\sup(A_\delta)=\delta$;
(2) $\{\gamma\in S\cap\delta\mid\exists i<\lambda\,[A_\delta\cap\gamma\subseteq^* B^i_\gamma]\}$ *is stationary in* δ.

Theorem 3.6. *Suppose that λ is a singular strong limit cardinal of uncountable cofinality and* $\{\mu<\lambda\mid\square^*_\mu\text{ holds}\}$ *is stationary in* λ.

For every stationary $S\subset\lambda^+$, $I[S;\lambda]=I[\lambda^+]\restriction\mathrm{Tr}(S)$.

Proof. By Fact 2.8, fix a strictly increasing and continuous sequence $\langle\mu_i\mid i<\mathrm{cf}(\lambda)\rangle$ of cardinals, converging to λ, and a sequence $\vec f=\langle f_\delta\mid\delta<\lambda^+\rangle$ that constitutes an aligned scale in $\prod_{i<\mathrm{cf}(\lambda)}\mu_i^+$. Clearly,

$$\Sigma:=\{i<\mathrm{cf}(\lambda)\mid\mathrm{cf}(\mu_i)<\mathrm{cf}(\lambda)<\mu_i\ \&\ \square^*_{\mu_i}\text{ holds}\}$$

is a stationary subset of $\mathrm{cf}(\lambda)$. For every $i\in\Sigma$, since $\square^*_{\mu_i}$ holds and μ_i is a singular cardinal, we may fix a matrix $\langle C^i_{\delta,j}\mid\delta<\mu_i^+,j<\mu_i\rangle$ such that for all $\delta<\mu_i^+$ and $j<\mu_i$:

(1) $C^i_{\delta,j}$ is a closed subset of δ with $\sup(C^i_{\delta,j})=\sup(\delta)$ and $\mathrm{otp}(C^i_{\delta,j})<\mu_i$;
(2) for every $\gamma\in\mathrm{acc}(C^i_{\delta,j})$, there exists $j'<\mu_i$ such that $C^i_{\delta,j}\cap\gamma=C^i_{\gamma,j'}$.

Claim 3.6.1. *Let $\delta<\lambda^+$. For every stationary $\Sigma'\subseteq\Sigma$ and every function* $g\in\prod_{i\in\Sigma'}\mu_i$*, there exists a stationary* $\Sigma''\subseteq\Sigma'$ *and a cardinal* $\mu<\lambda$ *such that, for every* $i\in\Sigma''$, $\max\{g(i),\mathrm{otp}(C^i_{f_\delta(i),g(i)})\}<\mu$.

Proof. By an application of Fodor's lemma. □

For all $\delta<\lambda^+$, a stationary $\Sigma'\subseteq\Sigma$ and $g\in\prod_{i\in\Sigma'}\mu_i$, let

$$A_{\delta,g}:=\{\beta\in E^\delta_{<\mathrm{cf}(\lambda)}\mid\sup\{i\in\mathrm{dom}(g)\mid f_\beta(i)\notin C^i_{f_\delta(i),g(i)}\}<\mathrm{cf}(\lambda)\}.$$

Claim 3.6.2. *Let $\delta < \lambda^+$, a stationary $\Sigma' \subseteq \Sigma$ and $g \in \prod_{i\in\Sigma'} \mu_i$.*

(1) *If $\mathrm{cf}(\delta) > \mathrm{cf}(\lambda)$ and δ is good, then $A_{\delta,g}$ is cofinal in δ;*
(2) $\mathrm{acc}^+(A_{\delta,g}) \cap E^{\delta}_{<\mathrm{cf}(\lambda)} \subseteq A_{\delta,g}$*;*
(3) $|A_{\delta,g}| < \lambda$.

Proof. (1) Suppose that $\mathrm{cf}(\delta) > \mathrm{cf}(\lambda)$ and that δ is good. Let $\epsilon < \delta$, and we will find an element of $A_{\delta,g}$ larger than ϵ. Recursively define a sequence of ordinals $\langle \alpha_n \mid n < \omega\rangle$ in δ, as follows. Let $\alpha_0 := \epsilon$. Next, given $n < \omega$ such that α_n has already been defined, we do the following. For all but boundedly many $i \in \Sigma'$, $f_{\alpha_n}(i) < f_\delta(i) = \sup(C^i_{f_\delta(i),g(i)})$, and then $\min(C^i_{f_\delta(i),g(i)} \setminus (f_{\alpha_n}(i)+1))$ is a well-defined element of $f_\delta(i)$. As $\vec{f}$ is aligned and δ is good, f_δ is an exact upper bound for $\vec{f} \restriction \delta$, so we may find $\alpha_{n+1} < \delta$ such that, for all but boundedly many $i \in \Sigma'$:

$$f_{\alpha_n}(i) < \min(C^i_{f_\delta(i),g(i)} \setminus (f_{\alpha_n}(i)+1)) < f_{\alpha_{n+1}}(i).$$

Put $\beta := \sup_{n<\omega} \alpha_n$, so that $\beta \in E^{\delta}_{<\mathrm{cf}(\lambda)}$. Then, for all but boundedly many $i \in \Sigma'$:

- $\langle f_{\alpha_n}(i) \mid n < \omega\rangle$ is a strictly increasing sequence converging to $f_\beta(i)$;
- $\langle \min(C^i_{f_\delta(i),g(i)} \setminus (f_{\alpha_n}(i)+1)) \mid n < \omega\rangle$ is a strictly increasing sequence converging to $f_\beta(i)$.

In particular, for all but boundedly many $i \in \Sigma'$, $f_\beta(i) \in \mathrm{acc}(C^i_{f_{\delta(i)},g(i)})$. Altogether, β is an element of $A_{\delta,g}$ above ϵ.

(2) Given $\gamma \in \mathrm{acc}^+(A_{\delta,g}) \cap E^{\delta}_{<\mathrm{cf}(\lambda)}$, we may find a cofinal subset $B \subseteq A_{\delta,g} \cap \gamma$ of order-type $\mathrm{cf}(\gamma)$ and a large enough $\varepsilon < \mathrm{cf}(\lambda)$ such that:

- for every $\beta \in B$, for every $i \in \Sigma' \setminus \varepsilon$, $f_\beta(i) \in C^i_{f_\delta(i),g(i)}$;
- for every $i \in \Sigma' \setminus \varepsilon$, $f_\gamma(i) = \sup_{\beta\in\beta} f_\beta(i)$.

Therefore, for every $i \in \Sigma' \setminus \varepsilon$, $f_\gamma(i) \in \mathrm{acc}(C^i_{f_{\delta(i)},g(i)})$. So, $\gamma \in A_{\delta,g}$.

(3) Use Claim 3.6.1 to fix a stationary $\Sigma'' \subseteq \Sigma'$ and a cardinal $\mu < \lambda$ such that, for every $i \in \Sigma''$, $\mathrm{otp}(C^i_{f_\delta(i),g(i)}) < \mu$. It follows that every element β of $A_{\delta,g}$ may be encoded by some function from Σ'' to μ, and hence $|A_{\delta,g}| \leq \mu^{\mathrm{cf}(\lambda)} < \lambda$. □

We are now in conditions to prove the theorem. Since λ is a strong limit, $I[\lambda^+;\lambda] = I[\lambda^+]$. So, by Remark 3.3, it suffices to prove that $I[S;\lambda] \supseteq I[\lambda^+] \restriction \mathrm{Tr}(S)$ for every stationary $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$. To this end, let $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ be stationary, and let T be a subset of $\mathrm{Tr}(S)$ lying in $I[\lambda^+]$; we shall show that $T \in I[S;\lambda]$, using Fact 3.5.

First, as $T \in I[\lambda^+]$, by [CFM04, Corollary 2.15], we may fix a club $D \subseteq \lambda^+$ such that every $\delta \in T \cap D$ is good for with respect to $\vec{f}$. Let $\vec{0}$ denote the constant function $g : \Sigma \to \{0\}$. By Claim 3.6.2, for every $\delta \in T \cap D$, $A_\delta := A_{\delta,\vec{0}}$ is a cofinal subset of δ of size $< \lambda$. In addition, for every $\delta \in T$, $S \cap \delta$ is stationary in δ. As $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$, it thus suffices to prove the following.

Claim 3.6.3. *Let $\gamma \in E^{\lambda^+}_{\mathrm{cf}(\lambda)}$. Then the following set has size no more than λ:*

$$\mathcal{B}_\gamma := \{A_\delta \cap \gamma \mid \delta \in T \cap D, \sup(A_\delta \cap \gamma) = \gamma\}.$$

Proof. Let $\delta \in T \cap D$ be such that $\sup(A_\delta \cap \gamma) = \gamma$. By Claim 3.6.2(2), we may fix a strictly increasing and continuous sequence $\langle \beta_\xi \mid \xi < \mathrm{cf}(\lambda)\rangle$ of ordinals in $A_\delta \cap \gamma$,

converging to γ. As $\vec{f}$ is aligned, we may fix a club Z in $\mathrm{cf}(\lambda)$, such that, for every $\zeta \in Z$,
$$f_\gamma(\zeta) = \sup_{\xi<\zeta} f_{\beta_\xi}(\zeta).$$
Recalling the definition of A_δ, for every $\xi < \mathrm{cf}(\lambda)$,
$$\sup\{i \in \Sigma \mid f_{\beta_\xi}(i) \notin C^i_{f_\delta(i),0}\} < \mathrm{cf}(\lambda).$$
So, by possibly shrinking Z, we may assume that, for every $\zeta \in Z$ and every $\xi < \zeta$,
$$\sup\{i \in \Sigma \mid f_{\beta_\xi}(i) \notin C^i_{f_\delta(i),0}\} < \zeta,$$
and in particular, if $\zeta \in \Sigma$, then $f_{\beta_\xi}(\zeta) \in C^\zeta_{f_\delta(\zeta),0}$.

Altogether, for every $i \in Z \cap \Sigma$, $f_\gamma(i) \in \mathrm{acc}(C^i_{f_\delta(i),0})$, so that, for some $g(i) < \mu_i$,
$$C^i_{f_\delta(i),0} \cap f_\gamma(i) = C^i_{f_\gamma(i),g(i)}.$$
It now follows from Claim 3.6.1 that there exists a stationary $\Sigma'' \subseteq Z \cap \Sigma$, a cardinal $\mu < \lambda$, and a function $g : \Sigma'' \to \mu$ such that, for every $i \in \Sigma''$, $C^i_{f_\delta(i),0} \cap f_\gamma(i) = C^i_{f_\gamma(i),g(i)}$. In particular, $A_\delta \cap \gamma \subseteq A_{\gamma,g}$ for such a g. So,
$$\mathcal{B}_\gamma \subseteq \bigcup\{\mathcal{P}(A_{\gamma,g}) \mid \Sigma'' \subseteq Z \cap \Sigma \text{ stationary}, \mu < \lambda, g : \Sigma'' \to \mu\}.$$
Recalling Claim 3.6.2(3) and the fact that λ is a strong limit, we infer that $|\mathcal{B}_\gamma| \leq \lambda$, as sought. $\square$

This completes the proof. $\square$

Fact 3.7 ([Rin10, Theorems 1 and 4]). *Suppose that λ is a singular cardinal, $S \subset \lambda^+$, and $I[S;\lambda]$ contains a stationary set. Then:*

- *$\mathrm{NS}_{\lambda^+} \restriction S$ is non-saturated;*
- *$2^\lambda = \lambda^+$ iff $\diamondsuit(S)$ holds.*

Corollary 3.8. *Suppose that λ is a singular strong limit cardinal of uncountable cofinality and $\{\mu < \lambda \mid \square^*_\mu \text{ holds}\}$ is stationary. Then $2^\lambda = \lambda^+$ iff $\diamondsuit(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$ holds.*

Proof. By Theorem 3.6, $I[E^{\lambda^+}_{\mathrm{cf}(\lambda)};\lambda] = I[\lambda^+] \restriction E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$. By [She93], $I[\lambda^+] \restriction E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ contains a stationary set. Now, appeal to Fact 3.7(2). $\square$

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